PHYSICAL REVIEW E **67**, 016611 (2003)

Dynamics and stability of solitary waves in optical-microwave interaction

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(Received 4 February 2002; revised manuscript received 21 October 2002; published 29 January 2003)

We study the dynamics and the stability of localized bound states of optical and microwave fields, which are linked together by a quadratic nonlinearity. The system is an example of an intense interaction between low and high frequency waves, as appears in many areas of physics. Perturbed solitary waves show a number of regular but damped oscillations with strong radiation from the microwave. It is demonstrated that these oscillations are caused by the excitation of several quasibound asymmetric linear modes of the solitary wave. The associated eigenvalues are found to be complex leading to a decay of the oscillations as observed numerically. Additional quasibound linear modes with a complex eigenvalue corresponding to exponential growth also exist, but due to physical constraints cannot be excited. Therefore, in contrast to systems solely with high frequency waves, the stability of the solutions is retained.

DOI: 10.1103/PhysRevE.67.016611 PACS number(s): 42.65.Tg, 42.65.Ky, 42.65.Sf

I. INTRODUCTION

Cascading effects between a fundamental and a second harmonic in media with a second-order nonlinearity are now a well understood topic [1]. In particular, the mutual locking between both waves has been a subject of intensive investigation (for a review, see for example, Ref. [2]). Here, phase modulation of the fundamental wave due to interaction with the second harmonic [3,4] leads to an effective third-order nonlinearity allowing propagation of bound states between fundamental and second harmonic, i.e., solitary waves. However, a less investigated effect in second-order nonlinearities which can also lead to cascading effects is optical rectification (OR) [5], meaning the generation of a static or quasistatic electric field from a coherent light source.

OR and the electro-optic effect (EOE) are known to connect the propagation of high frequency optical and quasi-dc microwave pulses in traveling wave structures, e.g., electro-optic modulators [6]. The resulting set of evolution equations [7,8] almost coincides with those derived to describe the interaction between short and long waves in hydrodynamics [9] or plasma physics [10]. Due to the distinct spectral properties of the two interacting waves, the mathematical expressions used to describe their evolution are extremely different. A real valued Korteweg de Vries (KdV) type equation accounts for the evolution of the long waves. In contrast, a slowly varying envelope approximation is applied to the high frequency components and a Schrödinger-like equation is obtained for the complex envelope of the optical field. The whole set of evolution equations allows for bright solitary wave solutions [11,12], where the joint action of microwave generation by OR and of the back coupling of the quasistatic electrical signal on the optical wave via the EOE mediates an effective cubic nonlinearity [13]. Here we show that a short-wave–long-wave interaction, where the wavelengths of the interacting waves differ by orders of magnitudes, gives rise to completely different and somehow unexpected dynamical effects. In particular, the absence of a gap in the linear spectrum of the microwave gives rise to an intriguing stable behavior.

In this work we first introduce the system of equations and discuss solitary wave solutions. Then we investigate their response on perturbations and perform a linear stability analysis.

II. EVOLUTION EQUATIONS AND SOLITARY WAVE SOLUTIONS

The mutual interaction of an optical with a microwave pulse in a traveling wave structure is governed by the following set of suitably scaled evolution equations [7,8]:

$$\left[\frac{\partial}{\partial z}+\delta\frac{\partial}{\partial t}-\sigma_{\mathrm{m}}\frac{\partial^3}{\partial t^3}\right]u_{\mathrm{m}}-\frac{\partial}{\partial t}|u_{\mathrm{o}}|^2=0, \tag{1}$$

$$\left[\mathrm{i}\frac{\partial}{\partial z}-\frac{\sigma_{\mathrm{o}}}{2}\frac{\partial^2}{\partial t^2}-u_{\mathrm{m}}\right]u_{\mathrm{o}}-\mathrm{i}\gamma\frac{\partial}{\partial t}[u_{\mathrm{o}}u_{\mathrm{m}}]=0. \tag{2}$$

The two interacting waves are described by completely different expressions because of their distinct spectral origin. u_{m} is the real valued amplitude of the microwave whose spectrum is centered around $\omega=0$. In contrast, the fast oscillating term of the high frequency optical component has already been removed and all the evolution is due to its complex slowly varying envelope u_{o}. The coordinate z corresponds to the propagation direction and t to the time in a reference frame moving with the speed of the optical pulse. The parameter δ describes the velocity mismatch between the waves. It causes linear microwaves to leave the optical pulse into a preferred direction and therefore to break the symmetry. $\sigma_{\mathrm{m}}=\pm 1$ and $\sigma_{\mathrm{o}}=\pm 1$ correspond to the sign of the dispersion of the microwave and the optical wave, respectively.

In the above set of equations we have scaled the most relevant nonlinear terms, i.e., the OR and the EOE, to unity.

Consequently the parameter γ, which ensures energy conservation, is normally extremely small ($\gamma < 10^{-3}$). Hence, energy losses of the optical wave due to the generation of quasistatic electrical fields are almost negligible. We have checked that this very small term has no qualitative and almost no quantitative effect on the field dynamics. Hence, it will be neglected by putting $\gamma = 0$ in the following.

For vanishing γ, respective conservation laws simplify considerably. The energy of the optical wave and something like an effective mass of the microwave, which are defined as

$$E_{\mathrm{opt}} = \int_{-\infty}^{\infty} |u_{\mathrm{o}}(z,t)|^2 dt$$

and

$$M_{\mathrm{m}} = \int_{-\infty}^{\infty} u_{\mathrm{m}}(z,t) dt, \tag{3}$$

are conserved. A third conservation law determines the motion of the center of gravity of the microwave,

$$S_{\mathrm{m}} = \frac{1}{M_{\mathrm{m}}} \int_{-\infty}^{\infty} t u_{\mathrm{m}}(z,t) dt \quad \text{as} \quad \frac{\partial}{\partial z} S_{\mathrm{m}} = \delta - \frac{E_{\mathrm{opt}}}{M_{\mathrm{m}}}. \tag{4}$$

Hence the center of gravity of the microwave moves always with a constant velocity, which is determined by the conserved quantities given above. In the absence of an optical excitation, the microwave components always travel into one direction, whereas a stationary field distribution requires a balance between effective mass of the microwave, energy of the optical component, and velocity mismatch.

Although the above system of Eqs. (1) and (2) is nonintegrable by means of the inverse scattering transform [14], there exists a two-parameter family of bright solitary waves. These are characterized by a propagation constant β and by a certain velocity. To simplify the analysis we restrict ourselves to stationary solutions with respect to the chosen reference frame. Moving solutions can be generated from resting ones by varying the velocity mismatch and using a simple transformation. In what follows we are looking for soliton solutions of the form

$$u_{\mathrm{m}}(z,t) = u_{\mathrm{ms}}(t), \tag{5}$$

$$u_{\mathrm{o}}(z,t) = u_{\mathrm{os}}(t) \exp(\mathrm{i}\beta z), \tag{6}$$

where β is the propagation constant. Introducing the ansatz Eqs. (5) and (6) into Eqs. (1) and (2) and integrating Eq. (1) once we obtain

$$\sigma_{\mathrm{m}} \frac{d^2}{dt^2} u_{\mathrm{ms}} - \delta u_{\mathrm{ms}} + |u_{\mathrm{os}}|^2 = 0, \tag{7}$$

$$\frac{\sigma_{\mathrm{o}}}{2} \frac{d^2}{dt^2} u_{\mathrm{os}} + \beta u_{\mathrm{os}} + u_{\mathrm{ms}} u_{\mathrm{os}} = 0. \tag{8}$$

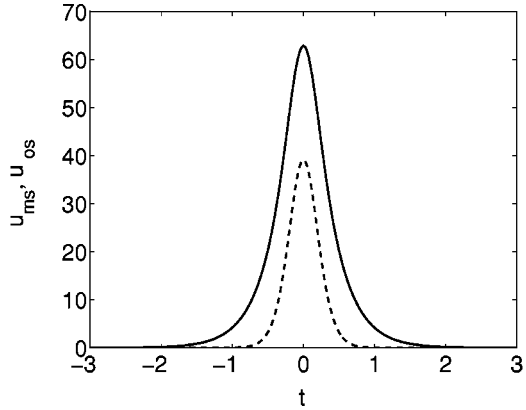

FIG. 1. Calculated amplitude profiles of solitary wave solution with $\delta = 10$, $\beta = -50$. Solid curve microwave u_{ms}; dashed curve optical wave u_{os}.

Note that for a real valued optical field u_{os} Eqs. (7) and (8) coincide with those obtained in second-harmonic generation (SHG) cascading [2,15], where the microwave plays a similar role as the second harmonic field. To allow for bright solitary wave solutions with evanescent tails the dispersion of the microwave σ_{m} and the velocity mismatch δ must have the same sign, $\sigma_{\mathrm{m}}\delta > 0$, which expresses the fact that the speed of the soliton must differ from any velocity of a linear microwave. Also, the wave number of the optical wave has to be distinct from that of any other linear wave of the optical spectrum. Consequently $\beta\sigma_{\mathrm{o}} < 0$ must hold. Here we restrict ourselves to overall normal dispersion and assume $\sigma_{\mathrm{o}} = \sigma_{\mathrm{m}} = 1$. It can be shown that for this choice of parameters, u_{ms} is always positive and therefore $M_{\mathrm{m}} > 0$ must hold.

Respective field profiles of solitary waves can be obtained for each propagation constant by solving Eqs. (7) and (8) numerically by means of a shooting technique [16]. Figure 1 shows the calculated amplitude profile of a solitary wave for the parameters $\delta = 10$, $\beta = -50$.

III. BEHAVIOR UNDER PERTURBATION

In the next step, we investigate the dynamical behavior of the solitary wave solutions. A solution of Eqs. (7) and (8) was determined numerically, perturbed and propagated according to Eqs. (1) and (2) by means of a Crank Nicholson scheme. The perturbation was performed in a similar manner to Ref. [17]. The aim was to keep the total power of the pulse constant while using an initial profile for the amplitudes $u_{\mathrm{m,o}}(z,t)$,

$$u_{\mathrm{m,o}}(0,t) = \left[u_{\mathrm{ms,os}}(t)^2 + \xi \frac{u_{\mathrm{ms,os}}^2(0)}{\frac{d^2}{dt^2} u_{\mathrm{ms,os}}^2(t)\big|_{t=0}} \frac{d^2}{dt^2} u_{\mathrm{ms,os}}^2(t) \right]^{1/2}, \tag{9}$$

where ξ represents the perturbation amplitude.

Figure 2 shows the outcome of a representative numerical experiment. The perturbed solitary wave shows quite regular oscillations around its stationary state. In contrast to, e.g., SHG solitons [17], there is a strong radiation from the mi-

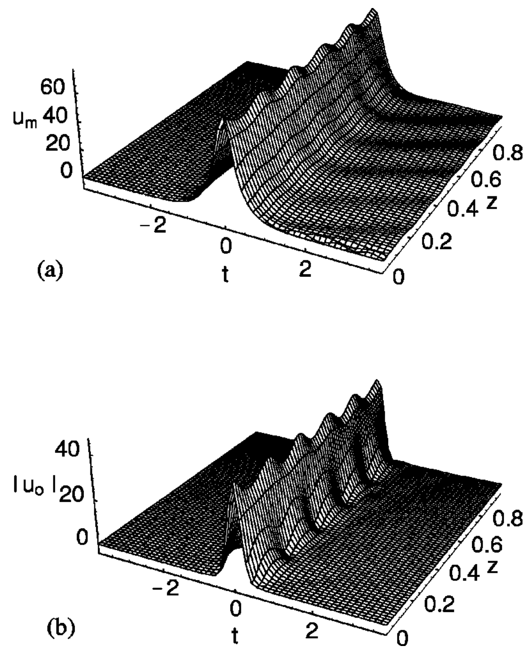

FIG. 2. Persistent oscillations of perturbed solitary wave solution; $\xi=0.2$, soliton parameters as in Fig. 1. (a) amplitude of the microwave u_{m}; (b) absolute value of the optical wave $|u_{\mathrm{o}}|$.

crowave and a certain damping of the oscillations is observed (see Fig. 3). Further numerical simulations showed that a number of discrete oscillation frequencies exists and even beating may occur. However, we never observed the underlying solitary wave to decay even for perturbations comparable with the soliton amplitude or for extremely long propagation distances $z\approx 5000$ (not shown here).

IV. LINEAR STABILITY ANALYSIS

It has been shown [17] that regular, long-lived oscillations of solitary waves correspond to internal eigenmodes or nontrivial, discrete bound states of the respective linearized problem. They are found in a large variety of nonintegrable systems, for example, the generalized KdV equation [18], the generalized nonlinear Schrödinger equation [19,20] or, as indicated above, the system describing SHG solitary waves [17].

To investigate the stability behavior and the influence of internal modes in more detail, we perform a linear stability analysis. To this end we assume the real valued soliton solution u_{ms}, u_{os} to be known and introduce small perturbations $\epsilon_{\mathrm{m,o}}(z,t)$ as $u_{\mathrm{m}}(z,t)=u_{\mathrm{ms}}(t)+\epsilon_{\mathrm{m}}(z,t)$ and $u_{\mathrm{o}}(z,t)=[u_{\mathrm{os}}(t)+\epsilon_{\mathrm{o}}(z,t)]\exp(i\beta z)$. Inserting this ansatz into Eqs. (1) and (2) results in the linearized set of equations,

$$\frac{\partial \epsilon_{\mathrm{m}}}{\partial z}+\delta\frac{\partial \epsilon_{\mathrm{m}}}{\partial t}-\frac{\partial^3 \epsilon_{\mathrm{m}}}{\partial t^3}-\frac{\partial}{\partial t}[u_{\mathrm{os}}(\epsilon_{\mathrm{o}}^*+\epsilon_{\mathrm{o}})]=0, \tag{10}$$

$$i\frac{\partial \epsilon_{\mathrm{o}}}{\partial z}-\beta\epsilon_{\mathrm{o}}-\frac{1}{2}\frac{\partial^2 \epsilon_{\mathrm{o}}}{\partial t^2}-(u_{\mathrm{os}}\epsilon_{\mathrm{m}}+u_{\mathrm{ms}}\epsilon_{\mathrm{o}})=0. \tag{11}$$

The small perturbations are expressed as

$$\epsilon_{\mathrm{m}}=\frac{1}{2}X_{\mathrm{m}}(t)\exp(\mathrm{i}\lambda z)+\frac{1}{2}X_{\mathrm{m}}^*(t)\exp(-\mathrm{i}\lambda^* z), \tag{12}$$

$$\epsilon_{\mathrm{o}}=\frac{1}{2}[X_{\mathrm{o}}(t)+Y_{\mathrm{o}}(t)]\exp(\mathrm{i}\lambda z)+\frac{1}{2}[X_{\mathrm{o}}(t)-Y_{\mathrm{o}}(t)]^*\exp(-\mathrm{i}\lambda^* z), \tag{13}$$

where $X_{\mathrm{m,o}}$ and Y_{o} refer to the in-phase and in-quadrature components of the perturbations. Note that the microwave is real and is perturbed by the in-phase component only. The following eigenvalue problem results:

$$L\mathbf{E}=\lambda\mathbf{E}, \tag{14}$$

where the eigenvector $\mathbf{E}(t)=[X_{\mathrm{m}}(t),X_{\mathrm{o}}(t),Y_{\mathrm{o}}(t)]^{\mathrm{T}}$. The operator L in the present case is given by

$$L=\begin{pmatrix} -i\left[\frac{d^3}{dt^3}-\delta\frac{d}{dt}\right] & -2i\left[u_{\mathrm{os}}\frac{d}{dt}+\frac{du_{\mathrm{os}}}{dt}\right] & 0 \\ 0 & 0 & -\frac{1}{2}\frac{d^2}{dt^2}-\beta-u_{\mathrm{ms}} \\ -u_{\mathrm{os}} & -\frac{1}{2}\frac{d^2}{dt^2}-\beta-u_{\mathrm{ms}} & 0 \end{pmatrix}. \tag{15}$$

For a solution $\mathbf{E}(t)$ with corresponding eigenvalue λ the vectors $\mathbf{E}^*(t)$, $\mathbf{E}(-t)$, and $\mathbf{E}^*(-t)$ are also solutions with eigenvalues $-\lambda^*$, $-\lambda$, and λ^*, respectively. Excitation of a perturbation with a negative imaginary eigenvalue component $\mathrm{Im}(\lambda)<0$ results in exponential growth, and hence is recognized as corresponding to an instability of the stationary solution. In contrast, a perturbation with a linear mode with positive imaginary part will eventually decay. The

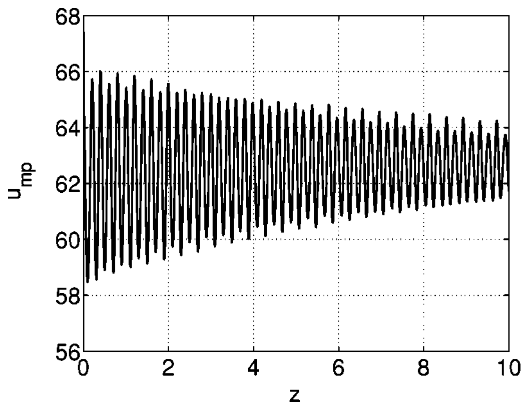

FIG. 3. Damped oscillation of perturbed soliton: peak amplitude of the microwave u_{mp}; all parameters as in Fig. 2.

eigensystem, Eq. (15), has two trivial eigensolutions at $\lambda = 0$,

$$\mathbf{E}=\begin{pmatrix} du_{\mathrm{ms}}/\mathrm{d}t \\ du_{\mathrm{os}}/\mathrm{d}t \\ 0 \end{pmatrix}, \tag{16}$$

which corresponds to a position shift in both waves and

$$\mathbf{E}=\begin{pmatrix} 0 \\ 0 \\ iu_{\mathrm{os}} \end{pmatrix}, \tag{17}$$

which corresponds to a phase shift in the optical wave. The asymptotic solutions for $|t|\to\infty$ of the optical perturbation functions $X_{\mathrm{o}}, Y_{\mathrm{o}}$ are obtained by setting u_{ms} and u_{os} in Eq. (15) to zero, giving

$$X_{\mathrm{o}}, Y_{\mathrm{o}} \propto \exp(\pm\Omega_{\mathrm{o}} t), \tag{18}$$

with

$$\Omega_{\mathrm{o}}=\sqrt{2(-\beta\pm\lambda)} \quad (X_{\mathrm{o}}=\mp Y_{\mathrm{o}}). \tag{19}$$

A similar consideration of the microwave perturbation function X_{m} gives $X_{\mathrm{m}}\propto\exp(\Omega_{\mathrm{m}}t)$ for $|t|\to\infty$ with three possible solutions for Ω_{m},

$$\Omega_{\mathrm{m}}=i(p_{-}-p_{+}), \tag{20}$$

$$\Omega_{\mathrm{m}}=\pm\frac{\sqrt{3}}{2}(p_{+}+p_{-})+\frac{1}{2}i(p_{+}-p_{-}), \tag{21}$$

where

$$p_{\pm}=\sqrt[3]{\pm\lambda/2+[(\delta/3)^3+(\lambda/2)^2]^{1/2}}. \tag{22}$$

Bound eigenstates of the optical field can exist for $\mathrm{Re}(\lambda)<|\beta|$. In contrast, for each real λ we find a solution with infinitely extended oscillatory tails of the microwave component. Hence, due to the third time derivative there is no gap in the continuum of the microwave.

This seems to contradict the requirements for the existence of bright solitary waves. Usually localized pulses exist in complete gaps of the linear spectrum. Hence, they have only evanescent waves available to form their low power tails. If this is not the case, stationary solutions can, in principle, couple to respective extended phase-matched waves. The consequence is the decay of a localized pulse due to the resulting energy losses. From this point of view the existence of localized solitary waves, which coexist with extended waves, seems to be an exceptional case. In fact, solitons which are located in the continuous spectrum have been found recently in systems with quadratic and cubic nonlinearities [21]. These so-called embedded solitons exist only for discrete wave numbers where the amplitude of extended waves vanishes leading to isolated (or zero family) soliton solutions. They were found to be unstable because energy reducing perturbations prevent the soliton from returning to its original wave number.

Although there is also no gap in the system discussed here, solitons belong to a two-parameter family and appear to be very robust. This surprising result is due to the particular properties of the KdV-type equation, which determines the shape of the microwave field. Because the equation of the microwave can be integrated once, the resulting expression (7) has a parabolic dispersion and allows only for evanescent linear waves provided that $\sigma_m\delta>0$. However, if we investigate dynamical properties or deal with the linearized problem and nonvanishing eigenvalues $\lambda\neq 0$ no integration can be performed and no gap appears. Equivalently, Eq. (20) suggests that for any real λ we will always find an eigensolution which possesses nondecaying and oscillating tails for $t\to\pm\infty$.

However, we still have to explain why only a few well-defined frequencies can be excited (see Fig. 2) although the spectrum of unbound linear states is continuous. There must be an additional constraint, which selects particular frequencies. The constraint is due to the physical requirement that the microwave can radiate in one direction only determined by the sign of the velocity mismatch δ. This is a consequence of the lowest order dispersion term containing the third derivative in Eq. (1). Figure 2 also demonstrates this constraint by the observed single-sided radiation. Hence, all linear waves, which have a nonzero amplitude at both sides $t\to\pm\infty$ cannot be excited. Only half-sided solutions of the linearized problem can influence the stability of the isolated solitary wave.

Therefore, we solve the eigenvalue problem Eq. (14) for a vanishing field at one boundary only by means of the Evans method in a similar fashion to Ref. [22]. In general, this corresponds to a scattering problem where we are particularly interested in the cases where the transmission vanishes. The validity of this technique was verified using a finite difference computation. As shown in Fig. 4, at discrete values of λ, half-sided eigenmodes with vanishing tails for either $t\to\infty$ or $t\to-\infty$ appear within the otherwise dense band of continuum modes. Interestingly, a large number of discrete modes can exist depending on the system parameters (see

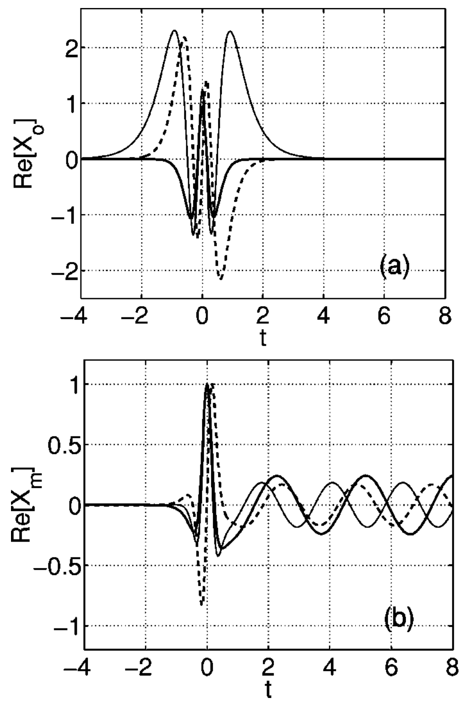

FIG. 4. Internal one-sided quasibound modes of the solitary wave depicted in Fig. 1; (a) real part of X_o; (b) real part of X_m. Thick solid curve $\mathrm{Re}(\lambda)=32.1$, dashed curve $\mathrm{Re}(\lambda)=43.0$, thin solid curve $\mathrm{Re}(\lambda)=48.2$.

below). To check if the one-sided internal modes are indeed the cause of the soliton oscillations in Fig. 2, a soliton solution was perturbed with a particular internal mode and subsequently propagated. As shown in Fig. 5, discrete oscillations can be excited separately and both the calculated eigenvalue and the oscillation frequencies are in good agreement, as seen in Table I.

The eigenvalues of these half-sided modes show additionally a small imaginary component $\mathrm{Im}(\lambda)>0$ (see Fig. 6) indicating a decay of respective perturbations. Hence, the energy loss due to the outflow of microwave power causes a decay of respective oscillations (see Fig. 3), while leaving

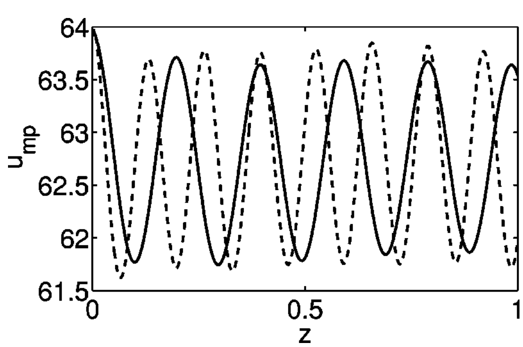

FIG. 5. Excitation of a soliton with its internal modes. The peak amplitude of the microwave is shown. Soliton parameters as in Fig. 4. Solid curve, excitation with an internal mode with $\mathrm{Re}(\lambda)=32.1$ (circular frequency of oscillation $\Omega=32.0$); dashed curve, excitation with an internal mode with $\mathrm{Re}(\lambda)=48.2$ (circular frequency of oscillation $\Omega=47.8$).

TABLE I. Oscillation frequencies and damping constants of perturbed solitary waves, $\beta=-50$. $\mathrm{Re}(\lambda)_{\mathrm{calc}}$ and $\mathrm{Im}(\lambda)_{\mathrm{calc}}$ are real and imaginary parts of calculated eigenvalue λ, $\mathrm{Re}(\lambda)_{\mathrm{prop}}$ and $\mathrm{Im}(\lambda)_{\mathrm{prop}}$ correspond to oscillation of soliton perturbed with internal mode.

δ	$\mathrm{Re}(\lambda)_{\mathrm{calc}}$	$\mathrm{Im}(\lambda)_{\mathrm{calc}}$	$\mathrm{Re}(\lambda)_{\mathrm{prop}}$	$\mathrm{Im}(\lambda)_{\mathrm{prop}}$
0.1	8.83	0.155	8.88	0.141
1	17.6	0.235	17.6	0.238
10	32.1	0.107	32.0	0.128

the soliton intact. As shown in Table I, the damping rate of internal oscillations is in good agreement with the imaginary part of the discrete eigenvalues $\mathrm{Im}(\lambda)$. Here, the soliton solutions were perturbed with particular modes in a similar manner to Fig. 5.

Considering the symmetry properties of the eigensystem discussed earlier, we also obtain a set of discrete solutions by reflection symmetry around $t=0$ with oscillating tails on the opposite side. They correspond to the physical situation that incoming waves are just absorbed by the soliton leading to a resonant excitation of oscillations. Therefore, respective eigenvalues have a negative imaginary part $\mathrm{Im}(\lambda)<0$, but play no role in the absence of an external source of radiation.

The location of all these different types of eigenvalues in the complex plane is sketched in Fig. 7. In domain (1) we find a dense band of radiation modes with both microwave and optical eigenfunctions are unbound. Domain (2) represents eigenmodes with bound optical and unbound microwave eigenfunctions. Domains (3) and (4) correspond to discrete right-sided and left-sided asymmetric perturbation functions, respectively.

It should be noted that eigenfunctions with oscillating tails or ''quasibound modes'' have also been found and proven to be responsible for soliton oscillations in the SHG system [17]. There the corresponding eigenvalue lies in the gap of the fundamental but in the continuum of the second harmonic.

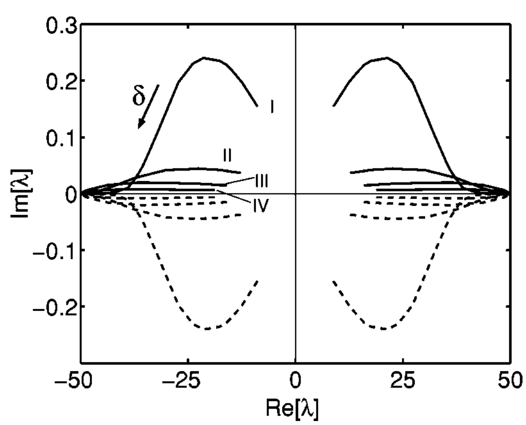

FIG. 6. Development of eigenvalues λ in complex plane with varying parameter δ ($\beta=-50$); solid curves, right-sided quasibound internal modes; dashed curves, left-sided quasibound modes. Depicted are the first four discrete modes.

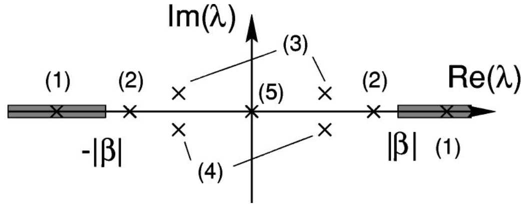

FIG. 7. Schematic structure of the spectrum of the linear eigenvalue problem Eqs. (14) and (15): (1) both microwave and optical eigenfunctions are unbound $[|\mathrm{Re}(\lambda)|>|\beta|,\ \mathrm{Im}(\lambda)=0]$, (2) bound optical and unbound microwave eigenfunctions (symmetric), $[0<|\mathrm{Re}(\lambda)|<|\beta|, \mathrm{Im}(\lambda)=0]$, (3) location of decaying quasibound modes $[\mathrm{Im}(\lambda)>0]$, (4) location of exponentially growing quasibound modes $[\mathrm{Im}(\lambda)<0]$, (5) trivial modes $(\lambda=0)$.

Figure 8 shows the calculated eigenvalues λ as a function of the parameter β and δ. For decreasing δ and hence decreasing velocity mismatch, a number of eigenvalues emerge from the boundary of the continuous spectrum of the optical wave. Note that in Fig. 8(b) only the eigenvalues of the first ten modes are shown. However, we did not find an upper limit for the number of half-sided modes. For large velocity mismatch, i.e., for large δ, there are no internal eigenmodes leading to a threshold for the appearance of soliton oscillations. A similar threshold exists in the case of SHG solitons [23] for the bifurcation of an internal mode from the continuous spectrum, where for large phase mismatch between fundamental and second harmonic the soliton does not support an internal mode.

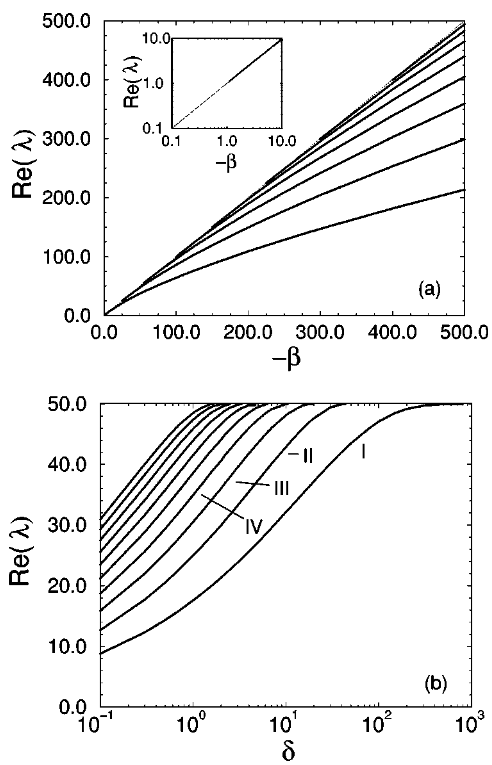

FIG. 8. Real part of eigenvalues Re(λ) of one-sided discrete internal modes as a function of soliton parameters. (a) $\delta=20$, the gap of the optical wave is marked by a dotted line; (b) eigenvalues of the first ten modes with $\beta=-50$; the numbering corresponds to eigenmodes in Fig. 6.

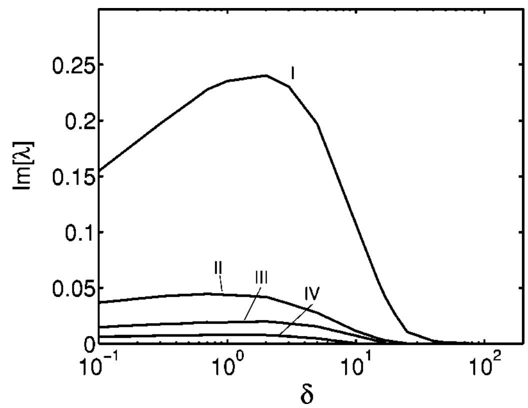

FIG. 9. Imaginary part of eigenvalues λ vs soliton parameter δ $(\beta=-50)$. Depicted are the first four discrete modes; the numbering corresponds to eigenmodes in Fig. 6.

As shown in Fig. 9, higher order modes are more weakly damped than lower order ones. This coincides with our observation that a symmetric perturbation corresponding to Eq. (9) leads to an initial strong excitation of the fundamental mode, whereas after a certain propagation length the dynamics is governed by a beating of oscillations with frequencies corresponding to higher order modes.

Even for strong perturbations, we never found any soliton to decay into linear radiation. A simple explanation of this robustness is based on the conservation law Eq. (4). For a solitary wave solution satisfying Eqs. (7) and (8), the center of gravity of the microwave is at rest. But, a decay into linear radiation would result in a constant flow of microwave energy into one direction. Hence, the center of gravity of the whole microwave distribution would shift eventually in contradiction to Eq. (4). Therefore, the main portion of the microwave component must remain localized and only waves with $M_{\mathrm{m}}=0$ can be emitted.

V. CONCLUSIONS

We have studied solitary waves in a system describing the interaction between a microwave and an optical wave in a second-order nonlinear medium. The equations are similar to a system of long-wave–short-wave interaction, which can be found in a variety of other physical systems. Perturbed solitons show persistent oscillations with strong radiation from the microwave. These oscillations stem from eigenmodes of the linearized problem. It is found that the solitary waves are always located in the continuum of the microwave. Apart from the dense band of continuum modes discrete half-sided modes with complex eigenvalues exist. As the microwave is physically constrained to radiate in one direction only, only modes which lead to regular, damped oscillations are excited. Whereas for large mismatch none of these degenerate states exist, a number of discrete eigenvalues bifurcate from the border of the continuous spectrum of the optical wave with decreasing velocity mismatch. No upper limit to the number of quasibound states seems to exist (see Fig. 8). Eigenstates which lead to soliton instability were not found. Soliton decay is prohibited by the physical conservation laws for the system.

ACKNOWLEDGMENTS

The German Research Foundation (DFG), the Engineering and Physical Sciences Research Council (EPSRC), and the Jane Draper endowment are thanked for financial support. We also acknowledge fruitful discussions with J. M. Arnold. K.B. thanks D. Michaelis and C. Etrich for valuable discussions and a warm hospitality during his stay at the Institute of Theoretical Optics, Friedrich Schiller University Jena.

[1] G.I. Stegeman, D.J. Hagan, and L. Torner, Opt. Quantum Electron. **28**, 1691 (1996).

[2] C. Etrich, F. Lederer, B.A. Malomed, T. Peschel, and U. Peschel, in *Progress in Optics*, edited by E. Wolf (Elsevier Science, Amsterdam, 2000), Vol. 41, pp. 485–565.

[3] H.J. Bakker, P.C.M. Planken, L. Kuipers, and A. Lagendijk, Phys. Rev. A **42**, 4085 (1990).

[4] A. Kobyakov and F. Lederer, Phys. Rev. A **54**, 3455 (1996).

[5] M. Bass, P.A. Franken, J.F. Ward, and G. Weinreich, Phys. Rev. Lett. **9**, 446 (1962).

[6] T.K. Gustafson, J.-P.E. Taran, P.L. Kelley, and R.Y. Chiao, Opt. Commun. **2**, 17 (1970).

[7] A.G. Kalocsai and J.W. Haus, Phys. Rev. E **52**, 3166 (1995).

[8] U. Peschel, K. Bubke, D.C. Hutchings, J.S. Aitchison, and J.M. Arnold, Phys. Rev. A **60**, 4918 (1999).

[9] T. Kawahara, N. Sugimoto, and T. Kakutani, J. Phys. Soc. Jpn. **39**, 1379 (1975).

[10] K.N. Nishikawa, H. Hojo, K. Mima, and H. Ikezi, Phys. Rev. Lett. **33**, 148 (1974).

[11] T. Yoshinaga and T. Kakutani, J. Phys. Soc. Jpn. **63**, 445 (1994).

[12] T. Yoshinaga, J. Phys. Soc. Jpn. **66**, 1018 (1997).

[13] C. Bosshard, R. Spreiter, M. Zgonik, and P. Günter, Phys. Rev. Lett. **74**, 2816 (1995).

[14] E. Benilov and S.P. Burtsev, Phys. Lett. **98A**, 256 (1983).

[15] A.V. Buryak and Y.S. Kivshar, Phys. Lett. A **197**, 407 (1995).

[16] U. Peschel, K. Bubke, D.C. Hutchings, and F. Lederer, Opt. Lett. **26**, 1090 (2001).

[17] C. Etrich, U. Peschel, F. Lederer, B. Malomed, and Y.S. Kivshar, Phys. Rev. E **54**, 4321 (1996).

[18] R.L. Pego and M.I. Weinstein, Philos. Trans. R. Soc. London, Ser. A **340**, 47 (1992).

[19] D.E. Pelinovsky, Y.S. Kivshar, and V.V. Afanasjev, Physica D **116**, 121 (1998).

[20] Y.S. Kivshar, D.E. Pelinovsky, T. Cretegny, and M. Peyrard, Phys. Rev. Lett. **80**, 5032 (1998).

[21] J. Yang, B.A. Malomed, and D.J. Kaup, Phys. Rev. Lett. **83**, 1958 (1999).

[22] D.C. Hutchings, J.M. Arnold, and D.F. Parker, Phys. Rev. E **58**, 6649 (1998).

[23] D.E. Pelinovsky, J.E. Sipe, and J. Yang, Phys. Rev. E **59**, 7250 (1999).